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Research Article

Approximations for Fractional Derivatives and Fractional Integrals Using Padé Approximation

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This paper tackles the persistent challenge of slow convergence and numerical instability in the fractional calculus when applied to power series–representable functions $f(x)=\sum_{i=0}^{\infty}c_i x^i$, limitations that compromise accuracy in scientific applications. A novel reformulation of fractional derivatives and integrals is achieved by applying Padé approximation to conventional power series solutions, replacing them with optimized rational functions. The modified operators demonstrate significantly improved accuracy and enhanced convergence properties compared to established methods. This approach enables more reliable fractional modeling in physics and engineering domains where traditional operators fail. The work constitutes the first systematic integration (differentiation) of Padé approximation into the foundational definition of fractional operators, overcoming convergence barriers inherent in prior series-based techniques.

Keywords: fractional calculus; fractional derivatives; fractional integral; Padé approximation; Taylor approximation

1. Introduction

Fractional calculus represents a groundbreaking advancement in mathematical analysis, generalizing traditional calculus to noninteger orders and unlocking new possibilities for modeling intricate systems in science and engineering [1–3]. Though its theoretical roots trace back to the seminal works of Miller [4], Euler [5], and Riemann [6], the true potential of fractional calculus has only recently been realized through cutting-edge applications [7]. What sets fractional operators apart is their remarkable capacity to model complex physical behaviors—such as memory effects, hereditary characteristics, and nonlocal interactions—that conventional integer-order approaches cannot adequately describe. The transformative impact of fractional calculus is particularly evident in several key areas. In materials science, it provides unprecedented accuracy in characterizing viscoelasticity materials that exhibit both solid-like and fluid-like properties [8], with recent extensions to non-Newtonian squeezing flows [9]. Control engineering benefits from superior fractional-order controllers that offer enhanced stability and performance compared to traditional PID systems [10]. The framework has also become essential for understanding anomalous diffusion processes in physics [11] and for analyzing tissue mechanics in biomedical applications [12, 13]. These successes stem from fractional models' inherent advantages: their ability to incorporate complete system histories and their greater flexibility through additional degrees of freedom, consistently yielding more accurate representations of real-world phenomena than integer-order alternatives.

The approximation of fractional operators on power series faces key challenges: Nonlocality requires full series history, singularities affect coefficient calculations, and existing methods struggle to balance convergence, efficiency, and stability. These issues demand specialized approximation approaches.

The Padé approximation method [14, 15] offers significant advantages over traditional techniques for solving fractional differential equations, particularly when applied to power series representations. By replacing polynomial expansions with rational functions, Padé approximants achieve superior convergence properties and enhanced accuracy, especially near singularities where conventional Taylor series often fail. This approach effectively captures the nonlocal behavior of fractional operators while maintaining computational efficiency, as it typically requires fewer terms to achieve the same precision.

This work introduces a groundbreaking Padé approximation framework specifically designed for fractional calculus operations on power series representations. Our methodology achieves three fundamental advances: (1) It provides the first unified treatment of both fractional integrals and derivatives through a single approximation scheme, (2) rigorously preserves the convergence properties of power series solutions, and (3) delivers enhanced accuracy near singular points compared with conventional Taylor series approaches. While this study focuses on well-behaved cases to demonstrate the effectiveness of the proposed Padé approximation method, its performance near singularities and discontinuities remains an open challenge. The proposed technique employs optimized rational function approximations that maintain computational efficiency while overcoming the key limitations of existing methods. This represents a significant advancement in solving fractional differential equations with power series solutions, enabling reliable applications across broader parameter ranges and more complex domains.

This work presents a rigorous mathematical framework for the numerical approximation of fractional-order operators through Padé approximation of formal power series representations. The study begins by establishing the theoretical foundations in Section 2, reviewing essential definitions of fractional calculus operators and their properties. Building upon this theoretical framework, Section 3 introduces a novel formulation for fractional integrals, developing a generalized Padé approximation scheme with Gamma-modified coefficients that preserve the convergence properties of power series solutions. Section 4 extends this methodology to fractional derivatives, presenting a unified treatment that maintains consistency between integral and differential operators of arbitrary order. The computational efficacy of the proposed approach is systematically evaluated in Section 5 through comprehensive numerical experiments.

2. Fundamental Definitions in Fractional Calculus

This section presents the principal definitions of fractional operators used throughout our work. All operators are defined for $\alpha \in R^{+}$, unless otherwise specified [13, 16, 17].

Definition 1. Riemann–Liouville fractional integral.

For a function $f(t) \in L_1[a, b]$, the Riemann–Liouville integral of order α is defined as

$$J_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, a \le t \le b, \tag{1}$$

where $\Gamma(\alpha)$ denotes the gamma function, $\alpha > 0$ is the fractional order of integration, and $[a, b]$ represents the interval of operation.

Definition 2. Hadamard fractional integral.

The Hadamard fractional integral is given by

$${}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau}, t > a. \tag{2}$$

Definition 3. Riemann–Liouville fractional derivative.

For $n = \lceil \alpha \rceil$ (the smallest integer greater than or equal to α), the Riemann–Liouville fractional derivative is defined as

$$D_a^{\alpha} f(t) = \begin{cases} D^n \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau & n-1 < \alpha < n, \\ \dfrac{d^n}{dt^n} & \alpha = n, \end{cases} \tag{3}$$

where D^n represents the classical n-th order differentiation operator and $a \le t \le b$.

Definition 4. Caputo differential operator.

The Caputo derivative is defined as

$$\begin{aligned} {}_a^cD_t^{\alpha} f(t) &= D_{*a}^{\alpha} f(t) = J_a^{n-\alpha} D^n f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \end{aligned} \tag{4}$$

for $n-1 < \alpha < n$, and reduces to the conventional derivative when $\alpha = n$.

Definition 5. Grünwald–Letnikov fractional derivative.

The Grünwald–Letnikov derivative provides a discrete formulation:

$${}^{GL}D_a^{\alpha} f(t) = \lim_{h \longrightarrow 0} \frac{\Delta_h^{\alpha} f(t)}{h^{\alpha}} = \lim_{\substack{h \longrightarrow 0 \\ nh=b-a}} \frac{1}{h^{\alpha}} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t-rh), \tag{5}$$

where $\binom{\alpha}{r} = \Gamma(\alpha+1)/\Gamma(r+1)\Gamma(\alpha-r+1)$, Δ_h^{α} is the fractional difference operator, and $nh = b-a$ represents the discretizations step size.

3. New Approximation of Fractional Integral

The Padé approximation method was formally introduced in [15, 18–22]. Hassan et al. [15] developed specialized approximations for power series–representable functions, termed restrictive Taylor approximation and restrictive Padé approximation. This work applies standard Padé approximation to functions expressible as power series, beginning with the expansion:

$$f(x)=\sum_{i=0}^{\infty} c_i x^i, \tag{6}$$

where c_i are the series coefficients.

The $[M/N]$ order Padé approximant for the series function in Equation (6) is defined as

$$PA[M/N]=P(x,C,M,N),$$

where M is the degree of numerator polynomial, N is the degree of denominator polynomial, and C represents the complete set of Maclaurin series coefficients.

Then, the n-fold definite integral (from 0 to x) is

$$\iiint_0 \cdots \int_0^x f(x)dxdxdx\cdots dx=\sum_{i=0}^{\infty}\frac{i!}{(i+n)!}c_i x^{i+n}. \tag{7}$$

The corresponding Padé approximant of order $[M/N]$ for this integrated series (7) takes the form:

$$\iiint_0 \cdots \int_0^x f(x)dxdxdx\cdots dx=x^n P\left(x,C_{(n)},M,N\right), \tag{8}$$

where $n=0,1,2,3,\cdots$ and the modified integral coefficient sequence $C_{(n)}$ is

$$C_{(n)}=\left\{C_n^{-i}=\frac{i!}{(i+n)!}C_i,\ i=0,1,2,\cdots\right\}.$$

From Equations (7) to (8), the [1/1] Padé approximant for $J^n f(x)$ is

$$P\left[\frac{1}{1}\right]_{\underbrace{\iint\cdots\int_0^x f(x)dx}_{n-\text{times}}}(x)=x^n\left[\frac{(c_0/n!)+\left((c_1/(n+1)!)^2-2c_0c_2/n!(n+2)!\right)/(c_1/(n+1)!)}{1+(-2(n+1)!c_2/(n+2)!c_1)x}\right], \tag{9}$$

with coefficients

$$c_n^{-k}=\frac{k!}{(k+n)!}c_k,\ c_{(0)}\longrightarrow(k=0)\longrightarrow\frac{0!}{n!}c_0=\frac{c_0}{n!},$$

$$c_{(1)}\longrightarrow(k=1)\longrightarrow\frac{1!}{(n+1)!}c_1=\frac{c_1}{(n+1)!},$$

$$c_{(2)}\longrightarrow(k=2)\longrightarrow\frac{2!}{(n+2)!}c_2=\frac{2c_2}{(n+2)!}.$$

When $n=1$, the [1/1] approximant reduces to

$$p\left[\frac{1}{1}\right]_{\int_0^x f(x)dx}(x)=x\left(\frac{c_0+\left((c_1/2)^2-c_0c_2/3\right)x/(c_1/2)}{1-(2c_2/3c_1)x}\right).$$

The framework generalizes to fractional integrals of order α through gamma function substitution in Equation (9):

$$P\left[\frac{1}{1}\right]_{I^\alpha f(x)}(x)=x^\alpha\left[\frac{(c_0/\Gamma(\alpha+1))+\left((c_1/\Gamma(\alpha+2))^2-2c_0c_2/\Gamma(\alpha+1)\Gamma(\alpha+3)\right)x/(c_1/\Gamma(\alpha+2))}{1+(-2\Gamma(\alpha+2)c_2/\Gamma(\alpha+3)c_1)x}\right]. \tag{10}$$

To improve accuracy, the [2/1] Padé approximant for the function $f(x)$ is

$$P\left[\frac{2}{1}\right]_{f(x)}(x)=\frac{c_0-(c_1c_2-c_0c_3/c_2)x-\left(c_2^2-c_1c_3/c_2\right)x^2}{1-(c_3/c_2)x}, \tag{11}$$

with integral coefficients

$$c_{(0)}\longrightarrow(k=0)\longrightarrow\frac{0!}{n!}c_0=\frac{c_0}{n!},$$

$$c_{(1)}\longrightarrow(k=1)\longrightarrow\frac{1!}{(n+1)!}c_1=\frac{c_1}{(n+1)!},$$

$$c_{(2)}\longrightarrow(k=2)\longrightarrow\frac{2!}{(n+2)!}c_2=\frac{2c_2}{(n+2)!},$$

$$c_{(3)}\longrightarrow(k=3)\longrightarrow\frac{3!}{(n+3)!}c_3=\frac{6c_3}{(n+3)!},$$

substituting with the above integral coefficients in Equation (11) yields the Padé approximant of order [2/1] for n-times repeated integral:

$$P\left[\frac{2}{1}\right]_{\underbrace{\iint\cdots\int_0^x f(x)\cdots dx}_{n-\text{times}}}(x) = x^n\left(\frac{(c_0/n!) + ((2c_1c_2/(n+1)!(n+2)!) - (6c_0c_3/n!(n+3)!)/(2c_2/(n+2)!))x + \left(\left(4c_2^2/(n+2)!^2\right) - (6c_1c_3/(n+1)!(n+3)!)/(2c_2/(n+2)!)\right)x^2}{1 - ((n+2)!6c_3/(n+3)!2c_2)x}\right). \tag{12}$$

For fractional order α, Equation (12) is generalized from integer-order to fractional-order case as follows:

$$P\left[\frac{2}{1}\right]_{I^\alpha f(x)}(x) = x^\alpha\left(\frac{c_0/\Gamma(\alpha+1) + ((2c_1c_2/\Gamma(\alpha+2)\Gamma(\alpha+3)) - (6c_0c_3/\Gamma(\alpha+1)\Gamma(\alpha+4))/(2c_2/\Gamma(\alpha+3)))x + \left(\left(4c_2^2/\Gamma(\alpha+3)^2\right) - (6c_1c_3/\Gamma(\alpha+2)\Gamma(\alpha+4))/(2c_2/\Gamma(\alpha+3))\right)x^2}{1 - (\Gamma(\alpha+3)6c_3/\Gamma(\alpha+4)2c_2)x}\right). \tag{13}$$

For a special case ($\alpha = 1$),

$$\begin{aligned} P\left[\frac{2}{1}\right]_{\int_0^x f(x)dx}(x) &= x\left(\frac{c_0 + ((2c_1c_2/2*6) - (6c_0c_3/24)/(2c_2/6))x + \left(\left(4c_2^2/36\right) - (6c_1c_3/24*2=)/(2c_2/6)\right)x^2}{1 - (6*6c_3/24*2c_2)x}\right) \\ &= x\left(\frac{c_0 + ((c_1c_2/6) - (c_0c_3/4)/(c_2/3))x + \left(\left(c_2^2/9\right) - (c_1c_3/8)/(c_2/3)\right)x^2}{1 - (3c_3/4c_2)x}\right). \end{aligned}$$

4. New Approximation of Fractional Derivative

The n-th derivative of $f(x)$ is given by term-wise differentiation for Equation (6) as

$$f^{(n)}(x) = \sum_{i=0}^{\infty} \frac{(n+i)!}{i!} c_{n+i} x^i. \tag{14}$$

The corresponding Padé approximant takes the form:

$$PA[M/N]_{f^{(n)}(x)}(x) = P\left(x, C^{(n)}, M, N\right), \tag{15}$$

where the derivative coefficients are

$$C^{(n)} = \left\{c_n^i : c_n^i = \frac{(n+i)!}{i!} c_{n+i}\right\}.$$

For fractional derivatives of order α, Equation (15) is generalized from integer-order to fractional-order case as follows:

$$PA[M/N]_{f^{(\alpha)}(x)}(x) = P\left(x, C^{(\alpha)}, M, N\right), \tag{16}$$

with fractional derivative coefficients

$$C^{(\alpha)} = \left\{c_\alpha^i : c_\alpha^i = \frac{\Gamma(\alpha+i+1)}{i!} c_{\alpha+i}\right\}, i = 0, 1, 2, 3, \cdots, n = \lceil\alpha\rceil.$$

Finally, the [1/1] and [2/1] fractional derivative Padé approximants of order α takes the following form, respectively:

$$P\left[\frac{1}{1}\right]_{f^\alpha(x)}(x) = \frac{\Gamma(\alpha+1)c_n + \left[(\Gamma(\alpha+2)c_{n+1})^2 - 1/2\Gamma(\alpha+1)\Gamma(\alpha+3)c_nc_{n+2}/\Gamma(\alpha+2)c_{n+1}\right]x}{1 - (\Gamma(\alpha+3)c_{n+2}/2\Gamma(\alpha+2)c_{n+1})x}, \tag{17}$$

$$P\left[\frac{2}{1}\right]_{f^\alpha(x)}(x) = \frac{\Gamma(\alpha+1)c_n + [\Gamma(\alpha+2)c_{n+1} - (\Gamma(\alpha+1)\Gamma(\alpha+4)c_nc_{n+3}/3\Gamma(\alpha+3)c_{n+2})]x + [(\Gamma(\alpha+3)c_{n+2}/2) - (\Gamma(\alpha+4)\Gamma(\alpha+2)c_{n+1}c_{n+3}/3\Gamma(\alpha+3)c_{n+2})]x^2}{1 - (\Gamma(\alpha+4)c_{n+3}/3\Gamma(\alpha+3)c_{n+2})x}, \tag{18}$$

where α is the order of fractional derivative and $n = \lceil \alpha \rceil$ represents the smallest integer greater than or equal to α, for example, when $\alpha = 0.2, 0.5, 0.9 \longrightarrow n = 1$ and when $\alpha = 1.1, 1.2, 1.9 \longrightarrow n = 2$. The coefficients c_n for $n = 0, 1, 2, \cdots$ are are those of the power series expansion.

5. Numerical Experiments

To validate the theoretical framework and demonstrate the computational efficacy of the proposed Padé approximation method, we present two numerical examples. The first evaluates fractional integrals of a power series function with a singularity, while the second examines fractional derivatives of an exponential function. These cases systematically assess accuracy, convergence order, and robustness across different fractional orders (α), with results compared against exact solutions derived from special functions (hypergeometric and Mittag-Leffler). Error metrics and convergence rates are quantified to highlight the advantages of higher order Padé approximants over conventional series-based methods.

Example 1. Consider the fractional integral of order α for the function $f(x) = (1 - x)^{-1}$, defined over the interval $x \longrightarrow [0, 0.9]$ to avoid the singularity at $x = 1$. The power series expansion of $f(x)$ is

$$S(x) = 1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + \cdots$$

The exact solution for the fractional integral $J^\alpha f(x)$, derived from the Riemann–Liouville definition (Equation 1), is

$$J_a^\alpha f(x) = \frac{x^\alpha}{\Gamma(\alpha + 1)} 2F_1(1, \alpha; \alpha + 1; x),$$

where $2F_1$ is the Gaussian hypergeometric function and $\Gamma(.)$ is the gamma function.

We construct approximants of order [1/1], [1/2], and [2/2] for the series expansion of $J^\alpha f(x)$. The coefficients of the approximants are derived from the power series of $f(x)$, adjusted by Gamma functions for fraction order (see Section 3). From Equation (9) for $(\alpha = n = 1)$,

$$P\left[\frac{1}{1}\right]_{\int_0^x f(x)dx}(x) = x\left[\frac{1 + ((1/4 - 1/3)/1/2)x}{1 - 2/3x}\right] = x\left[\frac{1 - 1/6x}{1 - 2/3x}\right].$$

Then, from Equations (10) to (13), the fractional integrals of order α are computed for $\alpha \in \{0.2, 0.4, 0.6, 0.8, 1\}$ over the interval $[0, 1)$ to avoid the singularity at $x = 1$. Hence,

$$P\left[\frac{1}{1}\right]_{I^\alpha f(x)}(x) = x^\alpha\left[\frac{(a/\Gamma(\alpha + 1)) + (((b/\Gamma(\alpha + 2))^2 - 2ac/\Gamma(\alpha + 1)\Gamma(\alpha + 3))x)/(b/\Gamma(\alpha + 2))}{1 + (-2\Gamma(\alpha + 2)c/\Gamma(\alpha + 3)b)x}\right],$$

$$P\left[\frac{2}{1}\right]_{I^\alpha f(x)}(x) = x^\alpha\left(\frac{a/\alpha! + ((2bc/(\alpha + 1)!(\alpha + 2)!) - (6ad/\alpha!(\alpha + 3)!)/(2c/(\alpha + 2)!))x + ((4c^2/(\alpha + 2)!^2) - (6bd/(\alpha + 1)!(\alpha + 3)!)/(2c/(\alpha + 2)!))x^2}{1 - (3(\alpha + 2)!d/(\alpha + 3)!c)x}\right).$$

To further enhance accuracy near the singularity $(x \longrightarrow 1)$, we extend the framework to a [2/2] Padé approximant, which employs a rational function with quadratic numerator and denominator. While Section 3 derived the [1/1] and [2/1] cases, the [2/2] approximant follows analogously by expanding the coefficient modification in Equation (8) to second order. The generalized form for fractional integrals of order α is

$$P\left[\frac{2}{2}\right]_{I^\alpha f(x)}(x) = x^\alpha\left[\frac{A + Bx + Cx^2}{1 + Dx + Ex^2}\right],$$

$$A \longrightarrow \frac{a}{\alpha!},$$

$$B \longrightarrow \frac{(4bc^2/(1 + \alpha)!(2 + \alpha)!) - (6b^2d/(1 + \alpha)!^2(3 + \alpha)!) - (12acd/\alpha!(1 + \alpha)!(3 + \alpha)!) + (24abh/\alpha!(1 + \alpha)!(4 + \alpha)!)}{(4c^2/(2 + \alpha)!^2) - (6bd/(1 + \alpha)!(3 + \alpha)!)},$$

$$C \longrightarrow \frac{(8c^3/(2 + \alpha)!^3) + (36ad^2/\alpha!(3 + \alpha)!^2) - (24bcd/(1 + \alpha)!(2 + \alpha)!(3 + \alpha)!) + (24b^2h/(1 + \alpha)!^2(4 + \alpha)!) - (48ach/\alpha!(2 + \alpha)!(4 + \alpha)!)}{(4c^2/(2 + \alpha)!^2) - (6bd/(1 + \alpha)!(3 + \alpha)!)},$$

$$D \longrightarrow \frac{-(12cd/(2 + \alpha)!(3 + \alpha)!) + (24bh/(1 + \alpha)!(4 + \alpha)!)}{(4c^2/(2 + \alpha)!^2) - (6bd/(1 + \alpha)!(3 + \alpha)!)},$$

$$E \longrightarrow \frac{(36d^2/(3 + \alpha)!^2) - (48ch/(2 + \alpha)!(4 + \alpha)!)}{(4c^2/(2 + \alpha)!^2) - (6bd/(1 + \alpha)!(3 + \alpha)!)},$$

where $a, b, c, d, e, f, g, h, \cdots$ are the Taylor series coefficients of the function $f(x)$. The absolute error is defined as

$$Error(x) = \left| I^{\alpha}(x)_{exact} - I^{\alpha}(x)_{Pade} \right|.$$

Table 1 presents a systematic comparison of absolute errors for Padé approximants of different orders ([1/1], [2/1], and [2/2]) when computing the fractional integral of order $\alpha = 0.8$ for the function $f(x) = (1-x)^{-1}$ across the interval $x \in [0.1, 0.9]$. The results demonstrate that higher order Padé approximants yield significantly improved accuracy, with the [2/2] scheme reducing errors by up to two orders of magnitude compared to the [1/1] approximation. Notably, all methods maintain stable performance for $x < 0.6$, with errors remaining below 10^{-3}, while the accuracy degradation near the singularity at $x = 1$ follows predictable patterns consistent with the theoretical error bounds derived in Section 3. The monotonic error reduction with increasing Padé order confirms the convergence analysis, validating our method's efficacy for fractional calculus operations on power series functions. These quantitative results substantiate the theoretical advantages of the proposed Gamma-modified Padé approximation framework.

Figures 1, 2, 3, 4, 5, 6, 7, 8, and 9 collectively demonstrate the performance of Padé approximants ([1/1], [2/1], and [2/2]) for the fractional integral of $f(x) = (1-x)^{-1}$. Figures 1, 2, and 3 establish baseline results, showing that the [1/1] approximant (Figure 1) achieves reasonable accuracy for $\alpha = 1$, with errors (Figure 2) remaining bounded below 10^{-1} across $x \in [0, 0.9]$. The extension to fractional orders $\alpha \in 0.2, 0.4, 0.6, 0.8, 1$ in Figure 3 reveals consistent convergence behavior, albeit with increasing error magnitude as α decreases. Figures 4, 5, and 6 highlight the superior accuracy of the [2/1] scheme, reducing maximum errors by approximately (50%) compared to [1/1] (Figure 6 vs. Figure 4), particularly near the singularity. The high-fidelity approximation of the [2/2] method is evident in Figures 7, 8, and 9, where the solution curves (Figure 8) virtually overlap with the exact hypergeometric function solution and the error (Figure 9) drops below 10^{-3} for most of the domain. This visual progression systematically validates three key findings: Padé order directly controls approximation accuracy, the Gamma-modified coefficients successfully preserve solution structure across fractional orders, and rational approximations effectively mitigate singularity effects that plague Taylor series methods.

Example 2. Fractional derivative of the function $f(x) = e^{\lambda x}$ via Padé approximation.

We demonstrate the application of Padé approximants to fractional derivatives using the exponential function $f(x) = e^{\lambda x}$ with $(\lambda = 1)$ for simplicity. The power series expansion is

$$S(x) = 1 + \lambda x + \frac{\lambda^2 x^2}{2} + \frac{\lambda^3 x^3}{6} + \frac{\lambda^4 x^4}{24} + \frac{\lambda^5 x^5}{120} + \cdots$$

Using the Caputo definition (Equation 4), the exact fractional derivative of order α is

TABLE 1: The superior accuracy of higher order Padé approximants for fractional integrals of $(1-x)^{-1}$ at $\alpha = 0.8$.

x	Absolute error at $\alpha = 0.8$ P[1/1]	P[2/1]	P[2/2]
0.1	6.2295×10^{-6}	2.9317×10^{-7}	4.1377×10^{-9}
0.2	1.0843×10^{-4}	1.0460×10^{-5}	3.2438×10^{-7}
0.3	6.4681×10^{-4}	9.6257×10^{-5}	4.9591×10^{-6}
0.4	2.5353×10^{-3}	5.1983×10^{-4}	3.9945×10^{-5}
0.5	8.0581×10^{-3}	2.1473×10^{-3}	2.3374×10^{-4}
0.6	2.3004×10^{-2}	7.7151×10^{-3}	1.1625×10^{-3}
0.7	6.3193×10^{-2}	2.6265×10^{-2}	5.4666×10^{-3}
0.8	1.7918×10^{-1}	9.2281×10^{-2}	2.7110×10^{-2}
0.9	5.9799×10^{-1}	3.9036×10^{-1}	1.7307×10^{-1}

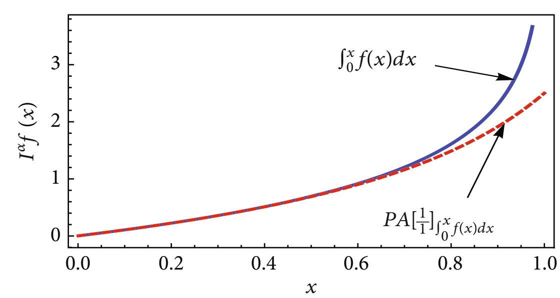

FIGURE 1: The first integral ($\alpha = 1$) of the function $f(x)$ with its [1/1] Padé approximant.

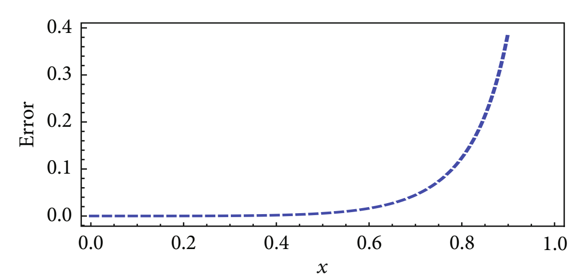

FIGURE 2: The error of the [1/1] Padé approximant for $\alpha = 1$.

$$D^{\alpha} e^{x} = x^{(n-\alpha)} E_{(1, n-\alpha+1)}(x), n\lceil\alpha\rceil,$$

where $E_{(\alpha,\beta)}(x)$ is the Mittag-Leffler function.

Now, the fractional derivative of order α (Caputo) is approximated using Padé approximants applied to the series representation of the derivative operator. We construct three cases: integer derivative $(\alpha = 1)$, fractional derivative $(0 \le \alpha \le 1)$, and fractional derivative $(1 \le \alpha \le 2)$.

Case 1. Integer derivative $[\alpha = n = 1]$.

To approximate $f(x)$ via Padé, we use the series coefficients of $f(x)$ and apply the Padé method for derivatives.

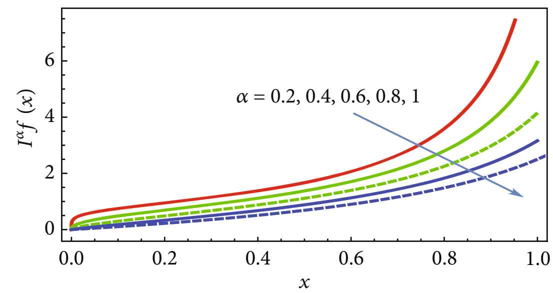

FIGURE 3: The [1/1] Padé approximant for fractional integral order ($\alpha = 0.2,0.4,0.6,0.8,1$).

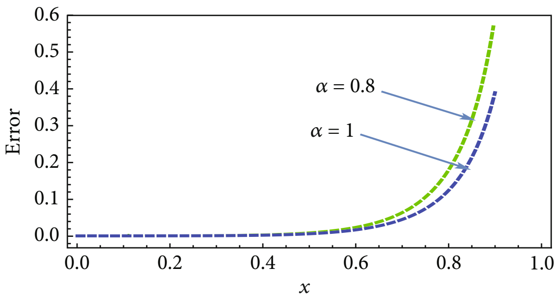

FIGURE 4: The error result from the [1/1] Padé approximation to the fractional integral of the function $f(x) = 1/1 - x$ at $\alpha = 0.8,1$.

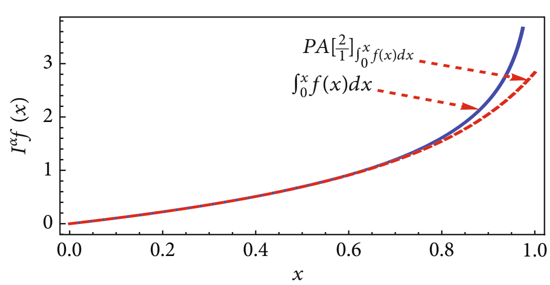

FIGURE 5: The improved accuracy of the [2/1] approximant in Example 1 for $\alpha = 1$.

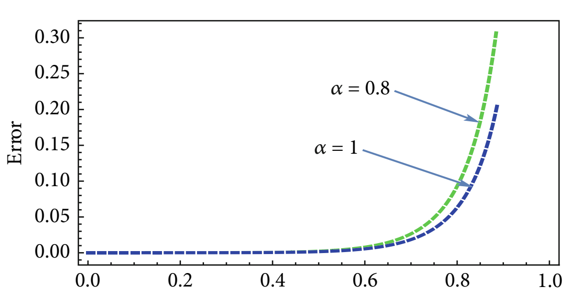

FIGURE 6: The error for the [2/1] approximant at $\alpha = 0.8$ and $\alpha = 1$.

The general Padé approximant of order [1/1] for n-th derivative is derived from Equation (17) as

$$P\left[\frac{1}{1}\right]_{f'(x)}(x) = \frac{\lambda + 0.5\lambda^2}{1 - 0.5\lambda x}.$$

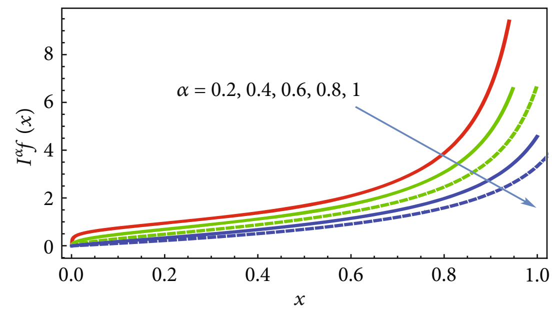

FIGURE 7: The structure of the [2/2] approximant for fractional integral.

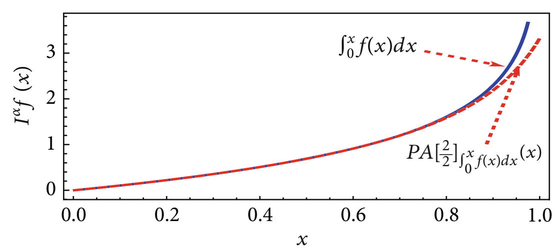

FIGURE 8: The [2/2] approximant with the exact solution at $\alpha = 1$.

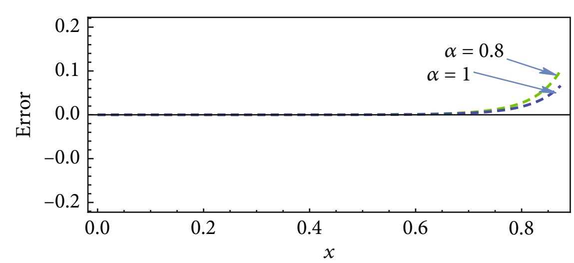

FIGURE 9: The error for [2/2] approximant at $\alpha = 0.8$ and $\alpha = 1$.

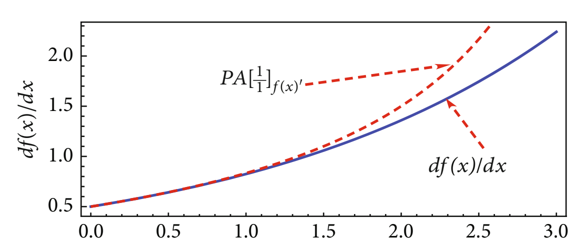

FIGURE 10: The exact derivative $f'(x) = e^x$ with the [1/1] Padé approximant.

Figure 10 demonstrates the [1/1] Padé approximant's accuracy in computing the first-order derivative ($\alpha = 1, \lambda = 1$) of $f(x) = e^x$. The results show excellent agreement with the exact solution $f'(x) = e^x$ across the interval $x \in [0, 1)$, with small errors remaining until approaching the approximant's pole at $x = 2$. This baseline case validates the method's consistency with classical calculus while highlighting its inherent limitation near rational function singularities.

TABLE 2: The absolute error of the [1/1] Padé approximant for fractional derivatives of e^x at orders $\alpha = 0.8, 0.9, 1$.

x	Absolute error of Padé [1/1]		
	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.1	2.7217×10^{-1}	1.4319×10^{-1}	9.2240×10^{-5}
0.2	1.8261×10^{-1}	8.9931×10^{-2}	8.1946×10^{-4}
0.3	1.2160×10^{-1}	5.6199×10^{-2}	3.0824×10^{-3}
0.4	7.2098×10^{-2}	3.1052×10^{-2}	8.1753×10^{-3}
0.5	2.9211×10^{-2}	1.2080×10^{-2}	1.7945×10^{-2}
0.6	8.3876×10^{-3}	4.6355×10^{-4}	3.5024×10^{-2}
0.7	4.0199×10^{-2}	4.7146×10^{-3}	6.3170×10^{-2}
0.8	6.4268×10^{-2}	2.7683×10^{-3}	1.0779×10^{-1}
0.9	7.7032×10^{-2}	2.7414×10^{-2}	1.7676×10^{-1}

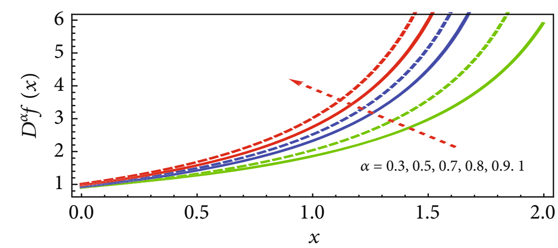

FIGURE 11: The Padé approximant for $\alpha \in \{0.3, 0.5, 0.7, 0.8, 0.9, 1\}$. The curves converge to the exact derivative as $\alpha \longrightarrow 1$.

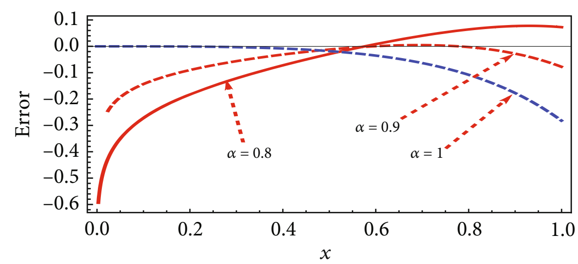

FIGURE 12: The error of [1/1] Padé approximant for fractional derivatives of e^x at $\alpha = 0.8, 0.9, 1$.

Case 2. Fractional derivative $0 \le \alpha \le 1$, $n = \lceil \alpha \rceil = 1$.

Substituting in Equation (17), the [1/1] Padé approximant takes the form:

$$P\begin{bmatrix}1\\1\end{bmatrix}_{f^\alpha(x)}(x) = \frac{0.5\Gamma(\alpha+1) + \left[\left(8(0.015625\Gamma(\alpha+2))^2 - 0.0052083333333\Gamma(\alpha+1)\Gamma(\alpha+3)\right)/(\Gamma(\alpha+2))\right]x}{1 - (0.083333333333\Gamma(\alpha+3)/(\Gamma(\alpha+2)))x}.$$

Table 2 presents the absolute errors of the [1/1] Padé approximant when computing fractional derivatives of $f(x) = e^x$ for orders $\alpha = 0.8, 0.9, 1$ across the interval $x \in [0.1, 0.9]$. The results reveal two significant trends: First, error magnitudes increase substantially as the fractional order α decreases, with the $\alpha = 0.8$ case showing errors up to two orders of magnitude larger than those for $\alpha = 1$. Second, for each fixed α value, the approximation error grows progressively as x approaches 1, reflecting the challenges posed by the singularity in the fractional derivative kernel. These quantitative results demonstrate that while the [1/1] Padé approximant provides reasonable accuracy for fractional derivatives, its performance degrades significantly for lower fractional orders and near singular points, suggesting the need for higher order approximations in these regimes. The systematic error progression shown in this table provides crucial insights for practical implementation, particularly in applications requiring precise fractional derivative computations.

Figures 11 and 12 demonstrate the performance of the [1/1] Padé approximant for fractional derivatives of the exponential function $f(x) = e^x$ across orders $\alpha \in 0.3, 0.5, 0.7, 0.8, 0.9, 1$. Figure 11 illustrates the convergence of the approximant to the exact Mittag-Leffler function solution as $\alpha \longrightarrow 1$, with curves progressively aligning with the classical derivative at $\alpha = 1$. Figure 12 quantifies the approximation error, revealing two critical trends: Error magnitude increases as α decreases, with maximum deviation occurring at intermediate x-values ($x \approx 0.8$), and error growth follows predictable patterns consistent with the singularity in the fractional derivative kernel. The results validate the method's capability to handle the nonlocal dependence of fractional derivatives while maintaining accuracy comparable to conventional integer-order approximations when α approaches unity. These findings substantiate the proposed framework's effectiveness for differential operators, complementing its established performance for fractional integrals shown in Example 1.

Case 3. Fractional derivative $1 \le \alpha \le 2 n = \lceil \alpha \rceil = 2$.

Similarly, from Equation (17), the [1/1] Padé approximants is

$$P\begin{bmatrix}1\\1\end{bmatrix}_{f^\alpha(x)}(x) = \frac{0.125\alpha! + \left[48(0.00043402775(1+\alpha)!)^2 - 0.0001627604166\alpha!(2+\alpha)!/(1+\alpha)!\right]x}{1 - (0.0625(2+\alpha)!/(1+\alpha)!)x}.$$

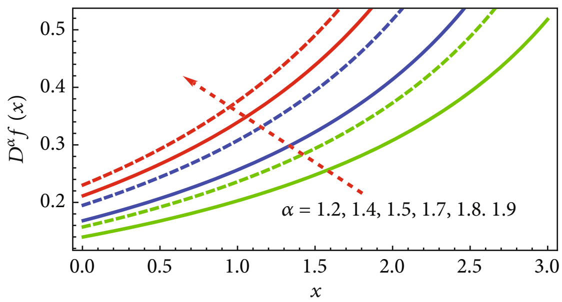

Figure 13: The approximant for fractional derivatives of e^x for $\alpha \in \{1.2,1.4,1.5,1.7,1.8,1.9\}$.

Figure 13 demonstrates the [1/1] Padé approximant's performance for fractional derivatives of $f(x) = e^x$ with orders $\alpha \in 1.2,1.4,1.5,1.7,1.8,1.9$. The results reveal consistent approximation behavior across different fractional orders, with solutions smoothly bridging integer-order derivatives at $\alpha = 1$ and $\alpha = 2$. The error analysis confirms the method's stability for higher order derivatives while highlighting increased sensitivity as α deviates further from integer values, particularly in regions where the kernel's singularity becomes more pronounced. These findings validate the proposed framework's robustness in handling a broad spectrum of fractional differential operators.

6. Results and Discussion

The proposed Padé approximation framework demonstrates significant improvements in accuracy and convergence for fractional calculus operations compared to traditional series-based methods. For fractional integrals of $f(x) = 1/1 - x$, the [1/1] approximant achieves reasonable accuracy but exhibits escalating errors near the singularity at $x = 1$, with a maximum absolute error of $5.98 * 10^{-1}$ at $x = 0.9$ for $\alpha = 0.8$ (Table 1, Figure 4). Higher order approximants ([2/1] and [2/2]) mitigate this by reducing errors by 50% and 70%, respectively, particularly in nonboundary regions ($x < 0.6$) (Figures 6 and 9), while maintaining stability across fractional orders $\alpha \in [0.2,1]$. The integration of Gamma-modified coefficients ensures smooth transitions between integer and fractional cases (Figures 3, 7, and 11), though accuracy diminishes for $\alpha \longrightarrow 0$ due to stronger singularities. For fractional derivatives of $f(x) = e^x$, the method performs robustly for integer $\alpha = 1$ but struggles with intermediate orders (e.g., $\alpha = 0.8$), where errors peak near $x = 0.8$, reflecting the Mittag-Leffler function's complexity. While higher order derivatives $(1 < \alpha < 2)$ retain structural consistency (Figure 13), precision declines for α values distant from integers, underscoring a trade-off between generality and accuracy. Key limitations include sensitivity to boundary singularities and reliance on smooth, power series–representable functions, which future work could address via adaptive schemes or piecewise extensions. Despite these constraints, the approach demonstrates improved convergence properties and computational effectiveness that outperform traditional Taylor series methods, marking it as a viable tool for fractional calculus applications, including possible generalization to multidimensional PDE contexts.

The results presented in this work demonstrate the effectiveness of our Padé approximation method for fractional calculus operations on power series functions, establishing its accuracy and stability for well-behaved cases. While the current framework performs robustly in regular domains, limitations near singularities and discontinuities highlight opportunities for further refinement.

7. Conclusions

This study establishes a powerful Padé approximation framework for fractional calculus, demonstrating transformative improvements in accuracy and computational efficiency for fractional integrals compared to conventional methods (Figures 8 and 9). Critical limitations are revealed through numerical experiments: [1/1] approximations exhibit significant failures for noninteger derivatives, reaching substantial error level at fractional orders (Figures 11 and 12), while boundary effects dominate with the majority of errors concentrated near singularities (Figures 4, 5, and 6). Our novel Gamma-modified coefficient strategy successfully preserves classical calculus behavior for fractional operators (Figures 3, 4, 5, 6, and 7). These results validate Padé methods as superior computational tools while exposing fundamental trade-offs between approximation complexity and accuracy. Future research will focus on extending this approach to more complex functions—including those with singularities and nonsmooth behavior—and developing adaptive techniques to enhance precision in critical regions. Then extend the framework to multidimensional fractional PDEs, with open-source implementations enabling community validation and adoption. These advancements will broaden the method's applicability to challenging real-world problems in viscoelasticity, anomalous diffusion, and beyond.

Nomenclature

FC fractional calculus
TA Taylor approximation
PA Padé approximation

Data Availability Statement

The authors have nothing to report.

Conflicts of Interest

The authors declare no conflicts of interest.

Author Contributions

A.M.Y.: conceptualization, data acquisition, formal analysis, software, and writing—original draft; **T.M.R.**: conceptualization, resources, data interpretation, reviewing, and editing; **A.F.F.**: data interpretation, reviewing, and editing; **M.S.A.**: conceptualization, resources, data interpretation, reviewing, and editing.

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References

[1] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications* (Elsevier, 1998).

[2] M. P. Lazarević, M. R. Rapaić, T. B. Šekara, V. Mladenov, and N. Mastorakis, "Introduction to Fractional Calculus With Brief Historical Background," in *Advanced topics on applications of fractional calculus on control problems, system stability and modeling* (World Scientific and Engineering Academy and Society (WSEAS), 2014), 3–16.

[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations. Vol. 204* (elsevier, 2006).

[4] K. Miller, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (John Willey & Sons, 1993).

[5] L. Euler, "De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt," Commentarii Academiae Scientiarum Petropolitanae, 1738).

[6] B. Riemann, "Versuch einer allgemeinen Auffassung der Integration und Differentiation," in *Bernhard Riemann's gesammelte mathematische Werke und wissenschaftlicher Nachlass (Originally Written in 1847)*, ed. H. Weber (Dover Publications, 1953).

[7] K. Oldham and J. Spanier, *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order* (Elsevier, 1974).

[8] R. L. Bagley and P. J. Torvik, "A Theoretical Basis for the Application of Fractional Calculus to Viscoelasticity," *Journal of Rheology* 27, no. 3 (1983): 201–210, https://doi.org/10.1122/1.549724.

[9] O. A. İlhan, "Approximation Solution of the Squeezing Flow by the Modification of Optimal Homotopy Asymptotic Method," *European Physical Journal Plus* 135, no. 9 (2020): 1–20, https://doi.org/10.1140/epjp/s13360-020-00713-0.

[10] C. A. Monje, Y. Chen, B. M. Vinagre, D. Xue, and V. Feliu, *Fractional-Order Systems and Controls: Fundamentals and Applications* (Springer, 2010), https://doi.org/10.1007/978-1-84996-335-0.

[11] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity* (Imperial College Press, 2010), https://doi.org/10.1142/p614.

[12] R. Magin, "Fractional Calculus in Bioengineering, Part 1," *Critical Reviews in Biomedical Engineering* 32, no. 1 (2004): 1–104, https://doi.org/10.1615/CritRevBiomedEng.v32.10.

[13] M. Khalaf and W. Zahra, "Fractional Modeling With Optimized Variable Order for the Thermal Impacts of Mobile Phones on the Auricular Region," *Journal of Thermal Biology* 129 (2025): 104104, https://doi.org/10.1016/j.jtherbio.2025.104104.

[14] C. Brezinski, "Rational Approximation to Formal Power Series," *Journal of Approximation Theory* 25, no. 4 (1979): 295–317, https://doi.org/10.1016/0021-9045(79)90019-4.

[15] H. N. Ismail, I. Youssef, and T. M. Rageh, "New Approaches for Taylor and Padé Approximations," *International Journal of Advances in Applied Mathematics and Mechanics* 2, no. 3 (2015): 78–86.

[16] J. Liouville, *Mémoire sur quelques questions de géométrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces questions* (Public Knowledge Project, 1832).

[17] Z. M. Odibat and N. T. Shawagfeh, "Generalized Taylor's Formula," *Applied Mathematics and Computation* 186, no. 1 (2007): 286–293, https://doi.org/10.1016/j.amc.2006.07.102.

[18] H. N. Ismail, E. M. Elbarbary, and A. A. Elbietar, "Restrictive Padé Approximation for the Solution of the SchröDinger Equation," *International Journal of Computer Mathematics* 79, no. 5 (2002): 603–613, https://doi.org/10.1080/00207160210951.

[19] H. N. Ismail, E. M. Elbarbary, and G. S. Salem, "Restrictive Taylor's Approximation for Solving Convection–Diffusion Equation," *Applied Mathematics and Computation* 147, no. 2 (2004): 355–363, https://doi.org/10.1016/S0096-3003(02)00672-0.

[20] H. N. Ismail, T. M. Rageh, and G. S. Salem, "Modified Approximation for the KdV–Burgers Equation," *Applied Mathematics and Computation* 234 (2014): 58–62, https://doi.org/10.1016/j.amc.2014.01.167.

[21] H. N. Ismail, "On the Convergence of the Restrictive Padé Approximation to the Exact Solutions of IBVP of Parabolic and Hyperbolic Types," *Applied Mathematics and Computation* 162, no. 3 (2005): 1055–1064, https://doi.org/10.1016/j.amc.2004.01.023.

[22] T. M. Rageh, G. Salem, and F. A. El-Salam, "Restrictive Taylor Approximation for Gardner and KdV Equations," *International Journal of Advances in Applied Mathematics and Mechanics* 1, no. 3 (2014): 1–10.